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Brane cosmology with an anisotropic bulk

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Abstract: In the context of brane cosmology, a scenario where our universe is a $3+1$-dimensional surface (the "brane") embedded in a five-dimensional spacetime (the "bulk"), we study geometries for which the brane is anisotropic — more specifically Bianchi I — though still homogeneous. We first obtain explicit vacuum bulk solutions with anisotropic three-dimensional spatial slices. The bulk is assumed to be empty but endowed with a negative cosmological constant. We then embed Z_2-symmetric branes in the anisotropic spacetimes and discuss the constraints on the brane energy-momentum tensor due to the five-dimensional anisotropic geometry. We show that if the bulk is static, an anisotropic brane cannot support a perfect fluid. However, we find that for some of our bulk solutions it is possible to embed a brane with a perfect fluid though its energy density and pressure are completely determined by the bulk geometry.

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1. Introduction

Since the advent of brane models with non-compact extra dimensions, brane world cosmology has been studied in depth (see [1]–[3] for reviews). The typical setup is one of a homogeneous and isotropic brane containing a perfect fluid, and the brane Friedmann equation can be obtained in (at least) two different ways. In the first approach [4, 5], coordinates are chosen relative to the brane which is therefore at a fixed position in the extra dimension. The bulk, on the contrary, is apparently time dependent and this time dependence induces, via the junction conditions, time dependence and hence cosmology on the brane. In the second approach [6, 7], the bulk is static and the brane moves along the extra dimension with this motion now being responsible for the cosmology. As was shown in [8], the symmetries (homogeneity and isotropy) of the brane impose that the static bulk is necessarily Sch-AdS. Furthermore these two approaches are completely equivalent via a coordinate transformation [9], and Birkhoff's theorem applies [8].

As in standard 4D cosmology, an obvious generalisation of such models is to consider homogeneous but anisotropic brane worlds [10]–[25]. Recall that in 4D, it is consistent with the Einstein equations to study a universe containing a perfect fluid [26]. (We use

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the standard definition of a perfect fluid for which there are no anisotropic stresses or energy fluxes so that $T^{\mu}_{\nu} = \mathrm{diag}(-\rho, P, P, P)$ with ρ the energy density and P the isotropic pressure.) In other words, geometric anisotropy is consistent with matter isotropy in 4D. Thus it is possible, for example, to study inflation due to a homogeneous scalar field and ask whether the universe isotropises. As proved by Wald [27], a positive cosmological constant will always isotropise the universe providing the perfect fluid matter satisfies the weak and strong energy conditions. Observational constraints on anisotropic universes are discussed in [28, 29].

The majority[1] of works on anisotropic brane world universes use the *effective* four-dimensional Einstein equations on the brane [31]

$$^4G_{ab} = \kappa_4^2 T_{ab} - \mathcal{E}_{ab} + \kappa^4 S_{ab}\,, \tag{1.1}$$

where $^4G_{ab}$ is constructed from the induced metric on the brane γ_{ab}, S_{ab} is quadratic in the brane stress energy tensor T_{ab}, and $\mathcal{E}_{ab}$ is the projection on the brane of the bulk Weyl tensor. Although this equation looks straightforward, it must be interpreted with care since the Weyl term not only depends on brane quantities — either geometry or matter — but also on the bulk metric. Thus the above brane equations are not closed. More precisely, the crucial unknown quantity in this effective approach is π^*_{ab}, the anisotropic component of $\mathcal{E}_{ab}$, defined in the decomposition [32]

$$-\frac{1}{\kappa_4^2}\mathcal{E}_{ab} = \rho^*\left(u_a u_b + \frac{1}{3}h_{ab}\right) + \pi^*_{ab} + q^*_{(a}u_{b)}\,. \tag{1.2}$$

Here u^a is a given 4-velocity, $h_{ab} = \gamma_{ab} + u_a u_b$ and ρ^*, $\rho^*/3$ and π^*_{ab} are the 'dark' energy density, pressure ($\mathcal{E}_{ab}$ is traceless so it acts as relativistic matter), and anisotropic stress respectively. The dark momentum density q^*_a vanishes for the Bianchi I branes considered here. The sign of $\rho^*/3$ is also crucial for Wald's theorem and brane isotropisation [15].

In the literature, in order to close the system of brane equations (1.1), various additional conditions on π^*_{ab} have been proposed in a rather ad hoc way. Whether or not the universe isotropises, or indeed whether or not inflation can take place at all depends on the choice made. Similar comments hold for the isotropy of the initial singularity.

In order to have a fully consistent picture, one cannot avoid specifying the bulk geometry. This has non trivial implications for the brane itself because, in brane world models, the Israel junction conditions — relating the extrinsic curvature to the matter on the brane — must be satisfied. For instance, suppose that the brane contains perfect fluid matter. Then the junction conditions will impose given relationships between the components of the extrinsic curvature: are these relations, together with the bulk Einstein equations, in fact compatible with an anisotropic brane? One of the initial motivations of this work was to try and answer that question.

A first step was taken in [17]. There the authors considered a moving brane in a static anisotropic bulk. Having solved the bulk Einstein equations, they then showed that the junction conditions induce anisotropic stresses in the matter on the brane. Hence the

[1] The exceptions, [16, 17], will be discussed below.

brane does not contain a perfect fluid. Furthermore, from the bulk Einstein equations and the junction conditions, it follows that this anisotropic stress can only vanish if the bulk is isotropic and hence the brane is isotropic! Thus, the conclusion is that "geometric anisotropy enforces, via the extrinsic curvature and the junction conditions, anisotropy of the matter fields" [17]. Finally, note that the anisotropic stresses obtained on the brane in [17] are *fixed* by the *bulk* metric. Thus, in general, they will not correspond to the intrinsic properties of physical matter.

Another attempt was made in [16]. For the choice of bulk metric made there, it was shown that the brane can only be anisotropic if it contains a constant tension, $\rho = P$!

In this paper we go one step further in the same direction. We first start by constructing new anisotropic solutions of the 5D bulk Einstein equations. The full five-dimensional Einstein equations are much more difficult to solve when one assumes 3D homogeneity *only*, rather than 3D homogeneity *and* isotropy as in the first works on brane cosmology. However, with some additional ansätze, we can find explicit geometries that generalize the solutions previously obtained in the literature.

We then introduce a brane in the anisotropic bulk geometries and study the constraints on the brane energy-momentum tensor. In particular, our aim is to see whether perfect fluid matter on the brane is compatible with non-zero geometric anisotropy. We show that a geometrically anisotropic brane moving in a *static bulk* necessarily has a stress energy tensor which is *not* that of a perfect fluid. Finally, we are able to construct a configuration in which matter on the brane is purely described by a perfect fluid, though its energy density and pressure as a function of brane time are completely determined by the geometry. As we will explain, this is not so surprising.

The paper is set up as follows. In section 2 we write down explicitly the full system of Einstein equations which we aim to solve. Then, in section 3, we solve them in two particular cases: firstly we assume that the anisotropy depends only on time and not on the extra-dimension; and secondly we assume that the metric components are *separable* with respect to time and to the extra-dimension. In section 4 we embed an anisotropic but homogeneous brane in the anisotropic bulk solutions and discuss whether or not the brane matter can be a perfect fluid. Conclusions are given in the final section.

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2. Bulk equations

We start from an ansatz for the bulk metric of the form

$$ds^2_{\text{bulk}} = -e^{2A_0(t,w)}dt^2 + \sum_{i=1}^{3} e^{2A_i(t,w)} \left(dx^i\right)^2 + dw^2 \,, \tag{2.1}$$

where the x^i coordinates span the three ordinary spatial dimensions and w is the coordinate of the extra dimension. Of course, one could start with a different choice of gauge, but we have found the assumptions $g_{ww} = 1$ and $g_{w\mu} = 0$ convenient for integrating the bulk Einstein equations.

In the bulk, endowed with a (negative) cosmological constant $\Lambda \equiv -6/\ell^2$, the five-dimensional Einstein's equations simply read

$$R_{AB} = \frac{2}{3}\Lambda g_{AB} \qquad (A = 0, \dots, 4)\,. \tag{2.2}$$

Inserting the metric ansatz (2.1) into Einstein's equations yields the following system of equations:

$$A_0'' + A_0' \sum_{\mu=0}^{3} A_\mu' + e^{-2A_0}\left[-\sum_{i=1}^{3} \ddot{A}_i - \sum_{i=1}^{3} \dot{A}_i^2 + \dot{A}_0 \sum_{i=1}^{3} \dot{A}_i\right] = \frac{4}{\ell^2} \tag{2.3}$$

$$A_i'' + A_i' \sum_{\mu=0}^{3} A_\mu' + e^{-2A_0}\left[-\ddot{A}_i - \dot{A}_i \sum_{j=1}^{3} \dot{A}_j + \dot{A}_0 \dot{A}_i\right] = \frac{4}{\ell^2} \tag{2.4}$$

$$\sum_{\mu=0}^{3} A_\mu'' + \sum_{\mu=0}^{3} {A_\mu'}^2 = \frac{4}{\ell^2} \tag{2.5}$$

$$\sum_{i=1}^{3} \dot{A}_i' - A_0' \sum_{i=1}^{3} \dot{A}_i + \sum_{i=1}^{3} \dot{A}_i A_i' = 0\,, \tag{2.6}$$

where a dash denotes a derivative with respect to w, and a dot one with respect to t. Note that middle of the alphabet latin indices label the three ordinary spatial directions in the bulk, whilst greek indices also include the time component. In order to solve these equations, it is convenient to introduce the average scale factor $s \equiv (\exp(\sum_i A_i))^{1/3}$ of the ordinary spatial directions x^i, and define

$$s \equiv e^A\,, \qquad A = \frac{1}{3}\sum_{i=1}^{3} A_i\,. \tag{2.7}$$

Furthermore we denote the deviation from isotropy by a vector $\mathbf{B}$ with components

$$B_i \equiv A_i - A\,, \tag{2.8}$$

so that

$$\sum_{i=1}^{3} B_i = 0\,. \tag{2.9}$$

Thus $\mathbf{B}'$ and $\dot{\mathbf{B}}$ quantify the spatial and temporal shear of the metric (2.1).

The Einstein equations can now be reexpressed in terms of the isotropic and anisotropic quantities we have just introduced. Eq. (2.3) becomes

$$A_0'' + A_0'\left(A_0' + 3A'\right) + e^{-2A_0}\left[-3\ddot{A} - 3\dot{A}^2 - \dot{\mathbf{B}}^2 + 3\dot{A}_0\dot{A}\right] = \frac{4}{\ell^2}\,, \tag{2.10}$$

while eq. (2.4) can be decomposed into an "isotropic" part, obtained by averaging over the index i,

$$A'' + A'\left(A_0' + 3A'\right) + e^{-2A_0}\left[-\ddot{A} - 3\dot{A}^2 + \dot{A}_0\dot{A}\right] = \frac{4}{\ell^2}\,, \tag{2.11}$$

and an "anisotropic" part that can be written as

$$\left[\mathbf{B}' e^{3A+A_0}\right]' = \left[\dot{\mathbf{B}} e^{3A-A_0}\right]^{\cdot} . \tag{2.12}$$

Finally, equations (2.5) and (2.6) give respectively

$$A_0'' + 3A'' + {A_0'}^2 + 3A'^2 + \mathbf{B}'^2 = \frac{4}{\ell^2} \tag{2.13}$$

and

$$\dot{A}' + \left(A' - A_0'\right)\dot{A} + \frac{1}{3}\dot{\mathbf{B}} \cdot \mathbf{B}' = 0 . \tag{2.14}$$

This rewriting of the equations easily allows us to see how anisotropy modifies the usual equations for the bulk: setting the anisotropic parts $\mathbf{B}$ to zero immediately yields the same system of equations as in isotropic brane cosmology [4, 5].

It is thus useful to try to generalise the integration of Einstein's equations in the context of isotropic brane cosmology to the present anisotropic case. A combination of (2.10), (2.11) and (2.13), in fact proportional to the component G_0^0 of the Einstein tensor, gives

$$A'' + 2A'^2 + \frac{1}{6}\mathbf{B}'^2 + e^{-2A_0}\left(-\dot{A}^2 + \frac{1}{6}\dot{\mathbf{B}}^2\right) = \frac{2}{\ell^2} . \tag{2.15}$$

Another combination of the same equations, corresponding to the component G_w^w of Einstein's tensor, gives

$$A'^2 + A_0' A' - \frac{1}{6}\mathbf{B}'^2 + e^{-2A_0}\left(-\ddot{A} - 2\dot{A}^2 + \dot{A}_0\dot{A} - \frac{1}{6}\dot{\mathbf{B}}^2\right) = \frac{2}{\ell^2} . \tag{2.16}$$

Following [5] let us now introduce the quantity

$$F \equiv e^{4A}\left(A'^2 - e^{-2A_0}\dot{A}^2\right) . \tag{2.17}$$

Then on using (2.6), one can rewrite (2.15), after multiplication by A', as

$$F' e^{-4A} + \frac{1}{3}\mathbf{B}'^2 A' + \frac{1}{3}e^{-2A_0}\left(\dot{\mathbf{B}}^2 A' - 2\dot{A}\,\dot{\mathbf{B}}.\mathbf{B}'\right) = \frac{4}{\ell^2}A' . \tag{2.18}$$

Similarly, (2.16) yields

$$\dot{F} e^{-4A} - \frac{1}{3}\mathbf{B}'^2\dot{A} - \frac{1}{3}e^{-2A_0}\dot{\mathbf{B}}^2\dot{A} + \frac{2}{3}A'\,\dot{\mathbf{B}}.\mathbf{B}' = \frac{4}{\ell^2}\dot{A} . \tag{2.19}$$

Note that when $\mathbf{B} = 0$, one recognises the results of isotropic brane cosmology [5] in which case the two above equations can readily be integrated. In the anisotropic case, these equations are much more difficult and, in order to integrate them, we will have to resort to simplifying assumptions as discussed in the next section.

3. Exact bulk solutions

In this section we make, in turn, two particular assumptions about the metric: these will enable us to integrate Einstein's equation explicitly. First we assume that the shear does not depend on the extra-dimension, namely that $\mathbf{B}' = 0$. Then we consider the situation in which the metric is separable, that is all the metric coefficients can be expressed as the product of a function of t with a function of w.

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3.1 Case $\mathbf{B}' = 0$

When the anisotropic parts are only time-dependent, i.e. $\mathbf{B}' = 0$, then the equations established in the previous section simplify greatly. This situation is similar to that studied in [33] in the case of Weyl metrics.

With $\mathbf{B}' = 0$, eq. (2.12) can be integrated to give

$$\dot{\mathbf{B}} = \boldsymbol{\lambda}(w) e^{A_0 - 3A}\,, \tag{3.1}$$

where $\boldsymbol{\lambda}(w)$ is an arbitrary function of w. On substituting into (2.18) and (2.19) we obtain

$$F' e^{-4A} + \frac{1}{3}\boldsymbol{\lambda}^2 e^{-6A} A' = \frac{4}{\ell^2} A' \tag{3.2}$$

and

$$\dot{F} e^{-4A} - \frac{1}{3}\boldsymbol{\lambda}^2 e^{-6A} \dot{A} = \frac{4}{\ell^2} \dot{A}\,. \tag{3.3}$$

Since $\boldsymbol{\lambda}$ is a function of w only, equation (3.3) can be integrated in time to yield

$$F + \frac{1}{6}\boldsymbol{\lambda}^2 e^{-2A} = \frac{1}{\ell^2} e^{4A} + \mathcal{C}(w)\,, \tag{3.4}$$

where $\mathcal{C}(w)$ is an arbitrary function of w. Substitution of this first integral in (3.2) gives a consistency relation between the integration functions $\boldsymbol{\lambda}(w)$ and $\mathcal{C}(w)$,

$$\frac{1}{3} e^{-2A} \left(\boldsymbol{\lambda}.\boldsymbol{\lambda}' - 2A'\boldsymbol{\lambda}^2\right) = \mathcal{C}' \tag{3.5}$$

or alternatively, using the relation $\boldsymbol{\lambda} = \dot{\mathbf{B}} e^{-A_0 + 3A}$,

$$e^{-2A}\left(A' - A_0'\right) = 3\frac{\mathcal{C}'(w)}{\boldsymbol{\lambda}^2(w)}\,. \tag{3.6}$$

Since $\boldsymbol{\lambda}$ and $\mathcal{C}$ are only w-dependent while A and A_0 in general are time-dependent, a simple way to satisfy this constraint is if both sides of (3.6) vanish. The more general case does not seem to yield more non-trivial solutions.

Assuming, therefore, that $\mathcal{C}$ is a constant and $A' = A_0'$, equation (2.14) implies $\dot{A}' = 0$ and therefore

$$A = \alpha(t) + \mathcal{A}(w), \quad A_0 = \alpha_0(t) + \mathcal{A}(w)\,. \tag{3.7}$$

This means that the metric is of the form

$$ds^2 = e^{2\mathcal{A}(w)}\left[-e^{2\alpha_0(t)} dt^2 + \sum_i e^{2\alpha_i(t)} {dx^i}^2\right] + dw^2 \tag{3.8}$$

$$\equiv a^2(w) h_{\mu\nu}(t) dx^\mu dx^\nu + dw^2\,, \tag{3.9}$$

which shows that the dependence on the extra-dimension reduces to a single warping factor. In this sense, the anisotropy will be the same on all slices $w =$ const, simply rescaled by this warping factor.

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We now solve explicitly for the warping factor $a(w) = \exp \mathcal{A}(w)$. From (2.5), or equivalently (2.13), it satisfies the equation

$$a'' = \frac{1}{\ell^2} a \,, \tag{3.10}$$

so that

$$a'^2 = \frac{1}{\ell^2} a^2 + c \,, \tag{3.11}$$

where c is an integration constant. Depending on the sign of c there are three solutions

$$a(w) = \begin{cases} \sqrt{c}\,\ell \sinh(w/\ell) & c > 0 \,, \\ \sqrt{-c}\,\ell \cosh(w/\ell) & c < 0 \,, \\ \exp(w/\ell) & c = 0 \,, \end{cases} \tag{3.12}$$

where we have absorbed another integration constant in a redefinition (by translation) of w, and w is also defined up to a sign. The other components of Einstein's equations can be rewritten in the form

$${}^{(5)}R_{\mu\nu} = {}^{(4)}R_{\mu\nu} - \left(aa'' + 3a'^2\right) h_{\mu\nu} = -\frac{4}{\ell^2} a^2 h_{\mu\nu} \,, \tag{3.13}$$

where ${}^{(4)}R_{\mu\nu}$ is the Ricci tensor for the four-dimensional metric $h_{\mu\nu}$. This implies

$${}^{(4)}R_{\mu\nu} = (3c) h_{\mu\nu} \tag{3.14}$$

so that $h_{\mu\nu}$ satisfies the 4D Einstein equations with cosmological constant $6c$.

We now turn to the time dependent functions $\alpha_i(t)$ in (3.8). Note that one can set $\alpha_0 = 0$ by a redefinition of the time coordinate, and this will be assumed below. For arbitrary c, let

$$\alpha_i(t) = \alpha(t) + \beta_i(t) \,, \tag{3.15}$$

where $\alpha(t) = \sum_i \alpha_i(t)/3$ and $\sum_i \beta_i(t) = 0$ in analogy with equations (2.7)–(2.9). Equation (3.1) can now be written in the form

$$\dot{\beta}_i = b_i e^{-3\alpha(t)} \,, \tag{3.16}$$

where the integration constants b_i satisfy $\sum_i b_i = 0$. The remaining Einstein equations reduce to

$$3\ddot{\alpha} + 3\dot{\alpha}^2 + \mathbf{b}^2 e^{-6\alpha} = 3c \qquad \text{and} \qquad \ddot{\alpha} + 3\dot{\alpha}^2 = 3c \tag{3.17}$$

as given by (3.14), so that

$$\dot{\alpha}^2 = c + \mathbf{b}^2 e^{-6\alpha}/6 \,. \tag{3.18}$$

On integrating it follows that, for $\mathbf{b}^2 \neq 0$,

$$e^{3\alpha(t)} = \begin{cases} \sqrt{\frac{\mathbf{b}^2}{6c}} \sinh(3\sqrt{c}t) & c > 0 \,, \\ \sqrt{\frac{\mathbf{b}^2}{6(-c)}} \sin(3\sqrt{-c}t) & c < 0 \,, \\ \sqrt{\frac{\mathbf{b}^2}{6}} (3t) & c = 0 \,, \end{cases} \tag{3.19}$$

and the β_i are given by (3.16)

$$\beta_i = \frac{\tilde{b}_i}{3} \begin{cases} \ln\tanh(\frac{3}{2}\sqrt{c}t) & c > 0\,, \\ \ln\tan(\frac{3}{2}\sqrt{-c}t) & c < 0\,, \\ \ln t & c = 0\,. \end{cases} \tag{3.20}$$

Here we have introduced a renormalised $\tilde{b}_i$, related to b_i by

$$\tilde{b}_i \equiv \sqrt{\frac{6}{\mathbf{b}^2}}\, b_i\,, \tag{3.21}$$

so that

$$\sum_i \tilde{b}_i = 0\,, \qquad \sum_i \tilde{b}_i^2 = 6\,. \tag{3.22}$$

Putting together the dependence on the extra-dimension (3.12) and the time dependence (3.19) and (3.20), the full metric in the simplest case when the effective cosmological constant on the brane c vanishes is given by

$$ds^2 = e^{2w/\ell}\left[-dt^2 + \sum_i t^{2p_i}\left(dx^i\right)^2\right] + dw^2 \tag{3.23}$$

with

$$\sum_i p_i = 1, \qquad \sum_i p_i^2 = 1\,. \tag{3.24}$$

We have replaced the $\tilde{b}_i$ by $p_i = (1+\tilde{b}_i)/3$, so as to recognise a warped version of the usual 4D Kasner metric [16].

When the effective cosmological constant is positive, $c > 0$, the full metric can be written, after some appropriate rescalings, in the form

$$ds^2 = \sinh^2\left(\frac{w}{\ell}\right)\left[-dt^2 + \sum_i \sinh^{2/3}\left(\frac{3t}{\ell}\right)\left(\tanh\left(\frac{3t}{2\ell}\right)\right)^{2\tilde{b}_i/3}\left(dx^i\right)^2\right] + dw^2\,. \tag{3.25}$$

Finally, the case of a negative cosmological constant is simply obtained by converting the hyperbolic functions into trigonometric ones.

3.2 Separable solution

Let us now go back to Einstein's equations in the form (2.3)–(2.6) and look for separable solutions

$$A_\mu(t,w) = \alpha_\mu(t) + \mathcal{A}_\mu(w)\,. \tag{3.26}$$

This time we do not assume that the spatial variation of the anisotropy, $\mathbf{B}'$, vanishes and the solutions of the previous subsection, which eventually turned out to be separable, are a priori only a subclass of the separable solutions.

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In the equations (2.3) and (2.4), we further assume that the brackets involving time derivatives separately vanish.[2] Leaving apart for now the mixed equation (2.6) the equations involving *spatial derivatives* reduce to the following system:

$$\mathcal{A}_\mu'' + \mathcal{A}_\mu' \sum_{\nu=0}^{3} \mathcal{A}_\nu' = \frac{4}{\ell^2}, \tag{3.27}$$

$$\sum_{\mu=0}^{3} \mathcal{A}_\mu'' + \sum_{\mu=0}^{3} \mathcal{A}_\mu'^{\,2} = \frac{4}{\ell^2}. \tag{3.28}$$

Although here we allow for a time dependence, these equations are exactly the same as those obtained from Einstein's equations when assuming a *static* ansatz, which is the situation studied in [17]. We can thus follow the method of [17] to integrate the *spatial* dependence of the metric. First introduce the quantity

$$u(w) = \exp\left(\sum_{\mu=0}^{3} \mathcal{A}_\mu\right). \tag{3.29}$$

Multiplying eq. (3.27) by u and summing over μ gives

$$u'' - \frac{16}{\ell^2} u = 0\,, \tag{3.30}$$

which admits the first integral,

$$u'^2 - \frac{16}{\ell^2}\left(u^2 + \gamma^2\right) = 0\,, \tag{3.31}$$

where we have written the integration constant as $-16\gamma^2/\ell^2$ for later convenience (we will also show that $\gamma^2 \geq 0$). One can then integrate (3.27) to obtain

$$\mathcal{A}_\mu' = \frac{1}{4}\frac{u'}{u} + \frac{4\gamma}{\ell}\frac{q_\mu}{u}\,, \tag{3.32}$$

where the q_μ are integration constants.

When $\gamma = 0$, equations (3.31) and (3.32) yield

$$\mathcal{A}_\mu(w) = \frac{w}{\ell}\,, \tag{3.33}$$

where the integration constants have been absorbed in a rescaling of the coordinates. This corresponds to the particular case $c = 0$ obtained in the previous subsection.

For arbitrary γ, the constants q_μ are constrained by eqs. (3.29), (3.28) and (3.27):

$$\sum_\mu q_\mu = 0, \qquad \sum_\mu q_\mu^2 = \frac{3}{4}\,, \tag{3.34}$$

[2] A priori, in the gauge $\alpha_0 = 0$, one could consider the more general case where the brackets are equal to four different constants. However, it can then be shown that these constants are necessarily equal, and that if the resulting single constant does not vanish then one recovers the situation considered in the previous section.

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implying $|q_\mu| \leq 3/4$. The second equality justifies the positivity of γ^2. Equation (3.31) then yields

$$u(w) = \gamma \sinh\left(\frac{4w}{\ell}\right), \tag{3.35}$$

where we have absorbed an integration constant in w. Note that replacing w by $-w$ also gives a valid solution. Thus eq. (3.32) gives

$$\mathcal{A}_\mu(w) = \frac{1}{4}\ln|u(w)| + \frac{4\gamma}{\ell} q_\mu v(w) + \text{const}, \tag{3.36}$$

where $v' = 1/u$, so that

$$v(w) = \frac{\ell}{4\gamma}\ln\tanh\left(\frac{2|w|}{\ell}\right). \tag{3.37}$$

Finally, we thus get

$$e^{2\mathcal{A}_\mu} = N_\mu^2 \sinh^{1/2}\left(\frac{4|w|}{\ell}\right)\tanh^{2q_\mu}\left(\frac{2|w|}{\ell}\right), \tag{3.38}$$

where N_μ are integration constants which must satisfy the constraint

$$\prod_{\mu=0}^{3} N_\mu^2 = \gamma^2, \tag{3.39}$$

following from eqs. (3.29) and (3.35). Note that replacing w by $-w$ in (3.35) would yield (3.38) with $-q_\mu$ in place of q_μ.

As mentioned above, when the metric is assumed to be *static*, the full Einstein equations reduce to the system (3.27)–(3.28) and the above results are enough to determine the full metric. Note that as discussed in [17], the case $(q_0, q_i) = (-3/4, 1/4)$ corresponds to Sch-AdS$_5$.

Here we assume that the metric is *not static* and thus the other Einstein equations must be integrated in order to determine the time dependence of the metric. Given the separable ansatz (3.26), it is easy to see that the time-dependent part of the metric components, imposing the gauge $\alpha_0 = 0$, must satisfy exactly the same equations as in section 3.1 with $c = 0$. The solutions are given in (3.19) and (3.20), with $c = 0$. However, there is also a further constraint coming from eq. (2.6) which relates time and space derivatives:

$$\sum_i q_i(\tilde{b}_i + 4) = 0. \tag{3.40}$$

Note that when all the q_i's are equal but non-zero, condition (3.40) is incompatible with (3.22) indicating that Sch-AdS$_5$ does not have a simple time dependent extension In general, however, constraints (3.22), (3.34) and (3.40) can all be satisfied simultaneously leading to time-dependent anisotropic solutions.

To summarize, the bulk solutions we have obtained are described by a metric of the form

$$ds^2 = \sinh^{1/2}\left(\frac{4w}{\ell}\right)\left[-\tanh^{2q_0}\left(\frac{2w}{\ell}\right)dt^2 + \sum_i \tanh^{2q_i}\left(\frac{2w}{\ell}\right)t^{2p_i}\left(dx^i\right)^2\right] + dw^2, \tag{3.41}$$

where the seven coefficients q_μ and p_i must satisfy the following five constraints

$$\sum_\mu q_\mu = 0\,, \qquad \sum_\mu q_\mu^2 = \frac{3}{4}\,, \qquad \sum_i p_i = 1\,, \qquad \sum_i p_i^2 = 1\,, \qquad \sum_i q_i\,(p_i+1) = 0\,. \tag{3.42}$$

One can solve explicitly this above system of constraints and, in appendix A, we give the general solution for the coefficients in terms of two parameters.

Interestingly, this metric "mixes" the five-dimensional static solution of [17] and the well-known four-dimensional Kasner solution. In this sense, we have found a much more sophisticated "warped" version of Kasner than in the previous subsection, because there are now four different warp factors, along the time and the three ordinary spatial directions.

As a final remark in this section, note that the above metric can be rewritten in the form

$$ds^2 = \sqrt{\frac{2z}{1-z^2}}\left[-z^{2q_0}dt^2 + \sum_i z^{2q_i}t^{2p_i}\left(dx^i\right)^2\right] + \frac{\ell^2}{4(1-z^2)^2}dz^2\,, \tag{3.43}$$

where $z = \tanh(2w/\ell)$. In appendix B, we have used this form to compute the square of the Weyl tensor, which is gauge-invariant and thus useful to analyse the physical singularities of the metric.

4. The brane

So far we have obtained explicit vacuum solutions for the bulk with a negative cosmological constant. In this section, we consider an infinitely thin brane embedded in anisotropic bulk geometries, and for simplicity study configurations with Z_2 symmetry about the brane. We start by establishing the general junction conditions, and then discuss the possibility of embedding a brane with only a perfect fluid as matter.

4.1 The embedding and junction conditions

Denoting the energy-momentum tensor of the brane matter as T^a_b, the Israel junction conditions, in the case of Z_2 symmetry, are given by

$$K^a_b = -\frac{\kappa^2}{2}\left(T^a_b - \frac{T}{3}\delta^a_b\right), \tag{4.1}$$

where K^a_b is the extrinsic curvature on one side of the brane. We use lowercase latin letters to denote the indices of the intrinsic coordinates on the brane. In general, the geometry of the brane can be defined by its embedding in the bulk space time, i.e. $X^A = X^A(x^a)$ where the x^a are the intrinsic brane coordinates. The extrinsic curvature is then given by

$$K_{ab} \equiv X^A_a X^B_b D_A n_B = \frac{1}{2}\left[g_{AB}(X^A_a\partial_b n^B + X^A_b\partial_a n^B) + X^A_a X^B_b n^C g_{AB,C}\right], \tag{4.2}$$

where D_A is the covariant derivative associated with the bulk metric g_{AB}, n^A is the unit vector normal to the brane, and $X^A_a \equiv \partial X^A/\partial x^a$.

If matter on the brane is a perfect fluid,

$$T^a_b = \mathrm{diag}(-\rho, P, P, P)\,, \tag{4.3}$$

then the junction conditions (4.1) imply that the spatial components of K^a_b must be equal

$$K^1_1 = K^2_2 = K^3_3 = -\frac{\kappa^2}{6}\rho\,, \tag{4.4}$$

whilst the off diagonal components vanish. These conditions must be satisfied at the brane position for all times.

Let us now introduce the bulk metric in the form

$$ds^2_{\mathrm{bulk}} = -e^{2A_0(t,w)}dt^2 + \sum_i e^{2A_i(t,w)}(dx^i)^2 + e^{2A_4(t,w)}dw^2\,, \tag{4.5}$$

where, in order to be more general, we have kept g_{ww} free. To obtain an anisotropic but *homogeneous* brane, we consider the embedding

$$X^A = (t_{\mathrm{b}}(\tau), \mathbf{x}, w_{\mathrm{b}}(\tau))\,, \tag{4.6}$$

where the subscript "b" stands for brane. The coordinates τ and $\mathbf{x}$ are the intrinsic brane coordinates. It is always possible to choose the time parameter τ to be the proper time by imposing the condition

$$e^{2A_0}\left(\frac{dt_{\mathrm{b}}}{d\tau}\right)^2 - e^{2A_4}\left(\frac{dw_{\mathrm{b}}}{d\tau}\right)^2 = 1\,. \tag{4.7}$$

As a consequence, the induced metric on the brane is simply given by

$$ds^2_{\mathrm{brane}} = -d\tau^2 + \sum_i e^{2A_i(\tau)}(dx^i)^2\,, \tag{4.8}$$

where $A_i(\tau) = A_i(t_{\mathrm{b}}(\tau), w_{\mathrm{b}}(\tau))$.

The shear in the bulk also induces a shear in the brane geometry, which can be expressed as $\sigma^a_b = \mathrm{diag}(0, \sigma^i)$, where

$$\sigma_i \equiv \frac{d}{d\tau}B_i(t_{\mathrm{b}}(\tau), w_{\mathrm{b}}(\tau)) = \dot{B}_i\Big|_X \frac{dt_b}{d\tau} + B'_i\big|_X \frac{dw_b}{d\tau}\,. \tag{4.9}$$

The brane is isotropic when the shear vanishes. Similarly, the average scale factor on the brane, H, is given by

$$H \equiv \frac{dA}{d\tau} = \dot{A}\Big|_X \frac{dt_b}{d\tau} + A'\big|_X \frac{dw_b}{d\tau}\,. \tag{4.10}$$

For the embedding (4.6) and metric (4.5) the normal to the brane, n^A, is

$$n^A = -\left(e^{-A_0+A_4}\dot{w}_{\mathrm{b}}, \mathbf{0}, e^{A_0-A_4}\dot{t}_{\mathrm{b}}\right) = -\left(e^{-A_0+A_4}\dot{w}_{\mathrm{b}}, \mathbf{0}, e^{-A_4}\sqrt{1+e^{2A_4}\dot{w}^2_{\mathrm{b}}}\right)\,, \tag{4.11}$$

where a dot on t_{b} and w_{b} denotes a derivation with respect to the *proper time* (and not the bulk time as for the bulk quantities) and where the second equality follows from the condition (4.7). Here we have chosen the normal with a negative w-component, meaning

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that we keep the bulk space-time with $w < w_b$. For our particular bulk solutions, this condition will lead to $\rho > 0$. Using the definition (4.2), one can then compute the spatial components of the extrinsic curvature tensor

$$K^i_0 = 0\,, \qquad K^i_j = 0 \quad (i \neq j)\,, \qquad K^i_i = \frac{1}{2g_{ii}}\left(n^t \dot{g}_{ii} + n^w g'_{ii}\right) = n^t \dot{A}_i + n^w A'_i\,. \tag{4.12}$$

The time/time component of the junction condition leads to the usual energy conservation equation on the brane, $d\rho/d\tau + 3H(\rho + P) = 0$, where H is the average Hubble parameter given in (4.10). This is the familiar result in the case of an empty bulk and follows from the Gauss equation and (4.1).

The junction conditions (4.1) together with the components of the extrinsic curvature tensor (4.12) imply that the brane energy-momentum tensor is necessarily of the form

$$T^a_b = \mathrm{diag}\,(-\rho, P_1, P_2, P_3)\,, \tag{4.13}$$

i.e. all the off-diagonal terms are necessarily zero. However, notice that this stress tensor is *generally anisotropic* with the pressure depending a priori on the direction, *unlike the perfect fluid form (4.3)*. Let P denote the isotropic pressure and decompose

$$P_i = P + \pi_i\,, \tag{4.14}$$

where the anisotropic stresses π_i satisfy $\sum_i \pi_i = 0$. It is then instructive to decompose the spatial components of the junction conditions (4.1) into an isotropic part,

$$e^{-A_0 + A_4} \dot{w}_b\, \dot{A}\Big|_X + e^{-A_4}\sqrt{1 + e^{2A_4}\dot{w}_b^2}\;\, A'\big|_X = \frac{\kappa^2}{6}\rho\,, \tag{4.15}$$

and an anisotropic part,

$$e^{-A_0 + A_4} \dot{w}_b\, \dot{B}_i\Big|_X + e^{-A_4}\sqrt{1 + e^{2A_4}\dot{w}_b^2}\;\, B'{}_i\big|_X = \frac{\kappa^2}{2}\pi_i\,. \tag{4.16}$$

This last relation tells us that, in general, an anisotropic bulk geometry implies the existence of an anisotropic stress on the brane.

4.2 Anisotropic brane with a perfect fluid?

We can now study the question of whether it is possible to find a bulk geometry and a brane trajectory such that (4.16) is satisfied with $\pi_i = 0$,

$$e^{-A_0 + A_4} \dot{w}_b\, \dot{B}_i\Big|_X + e^{-A_4}\sqrt{1 + e^{2A_4}\dot{w}_b^2}\;\, B'{}_i\big|_X = 0\,, \tag{4.17}$$

that is, for a *perfect fluid*. In order to do this, we consider in turn the bulk geometries constructed previously, as well as the case of a static bulk geometry.

4.2.1 A no-go condition: a static bulk and a moving brane

First consider a static anisotropic bulk, described by the metric

$$ds^2_{\rm bulk} = -e^{2A_0(w)}dt^2 + \sum_i e^{2A_i(w)}(dx^i)^2 + e^{2A_4(w)}dw^2 \,. \tag{4.18}$$

Since we are interested in cosmology, the brane is necessarily moving in this bulk, $\dot{w}_{\rm b} \neq 0$. As the bulk is static, $\dot{\mathbf{B}} = 0$, and condition (4.17) therefore reduces to

$$\mathbf{B}'\big|_X = 0 \,. \tag{4.19}$$

However, from (4.9), this implies that the shear on the brane also vanishes and the brane is isotropic.

Thus we conclude that it is not possible to have an anisotropic moving brane containing a perfect fluid in a static background of the form given in (4.18). Notice that in order to reach this conclusion it was not necessary to solve the bulk Einstein equations: a moving brane containing a perfect fluid in a static bulk is necessarily geometrically isotropic, and the bulk is therefore Sch-AdS$_5$ [8].

Conversely, if one embeds a moving brane into such a static anisotropic background then the brane stress energy tensor is not that of a perfect fluid: from the junction conditions, the stress energy tensor picks up a bulk-dependent anisotropic stress as in [17]. In other words, the matter on the brane is fixed by the bulk geometry with the anisotropic stress on the brane, $\pi_i \propto B_i'|_X$.

4.2.2 Bulk with $\mathbf{B}' = 0$

We now consider a bulk characterized by $\mathbf{B}' = 0$, as in section 3.1. Condition (4.17) then reduces to

$$\dot{w}_{\rm b}\, \dot{\mathbf{B}}\Big|_X = \mathbf{0} \tag{4.20}$$

so that there can only be a non-zero shear on the brane (see (4.9)) if the brane is at a fixed position in the extra dimension,

$$w_{\rm b} = \text{constant} \,. \tag{4.21}$$

As discussed in section 3.1, when $\mathbf{B}' = 0$ the bulk Einstein equations impose that the bulk metric is separable (see (3.7)). This implies that condition (4.15) reduces to the time independent expression

$$\rho = A'(w_b)\frac{6}{\kappa^2} \,, \tag{4.22}$$

because A' depends only on w and not t. The brane can therefore contain a perfect fluid but it must be a constant tension given by (4.22). The geometry on the brane is anisotropic — on choosing w_b such that $a^2(w_b) = 1$,

$$ds^2_{\rm brane} = -d\tau^2 + e^{2\alpha(\tau)} \sum_i e^{2\beta_i(\tau)}(dx^i)^2 \,, \tag{4.23}$$

where $\alpha(\tau)$ and $\beta_i(\tau)$ are given in (3.19) and (3.20). Finally, note that for these solutions the projected bulk anisotropic stress on the brane, π^*_{ab}, vanishes since it is proportional to $\mathbf{B}''|_{w_b}$.

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4.2.3 Separable bulk solution

We now assume that the bulk geometry is given by the metric (3.41), that is $A_4 = 0$ and

$$e^{A_0} = \sinh^{1/4}\left(\frac{4w}{\ell}\right)\tanh^{q_0}\left(\frac{2w}{\ell}\right), \tag{4.24}$$

$$e^{A} = \sinh^{1/4}\left(\frac{4w}{\ell}\right)\tanh^{-q_0/3}\left(\frac{2w}{\ell}\right)t^{1/3}, \tag{4.25}$$

$$e^{B_i} = \tanh^{q_i+q_0/3}\left(\frac{2w}{\ell}\right)t^{p_i-1/3}. \tag{4.26}$$

Expression (4.17) summarises three equations, one for each value of i which, for this bulk metric, yield

$$\frac{e^{-A_0}\,\dot{w}_{\rm b}}{\sqrt{1+\dot{w}_{\rm b}^2}} = -\left.\frac{B_i'}{\dot{B}_i}\right|_X = -\frac{4}{\ell}\left(\frac{q_i+q_0/3}{p_i-1/3}\right)\frac{t_{\rm b}}{\sinh(4w_{\rm b}/\ell)}. \tag{4.27}$$

Clearly the three relations in (4.27) are only compatible if the right-hand side is independent of i. Thus the coefficients p_i and q_i must satisfy the relation

$$q_i + \frac{q_0}{3} = k\left(p_i - \frac{1}{3}\right), \tag{4.28}$$

where k is a constant. Combining this with the constraints on p_i and q_i given in (3.42), or using the results of appendix A, implies

$$q_0 = \pm\frac{\sqrt{3}}{4} \tag{4.29}$$

and thus $k = \pm\sqrt{3}/2$. Remarkably, therefore, the brane *can support perfect fluid type matter* for $q_0 = \pm\frac{\sqrt{3}}{4}$. As shown explicitly in appendix A, it is possible to find sets of coefficients (p_i, q_i) that satisfy all the constraints plus the additional condition $q_0 = \pm\sqrt{3}/4$. All the possible solutions are expressed in terms of a single parameter.

Of course, the brane cannot support *any* perfect fluid, in contrast with the isotropic case. The reason is that the anisotropic junction condition (4.17) determines the trajectory of the brane. Indeed, on substituting $k = \pm\sqrt{3}/2$, this relation now becomes

$$\frac{\dot{w}_{\rm b}}{\sqrt{1+\dot{w}_{\rm b}^2}} = \mp\frac{2\sqrt{3}}{\ell}t_b\frac{e^{A_0}}{\sinh(4w_b/\ell)} \equiv f(t_b, w_b). \tag{4.30}$$

Thus, combining this with the expression for $\dot{t}_b$, one finds that the brane trajectory in spacetime is determined by integrating the first-order system

$$\frac{dt_b}{d\tau} = \frac{e^{-A_0}}{\sqrt{1-f^2}}, \qquad \frac{dw_b}{d\tau} = \frac{f}{\sqrt{1-f^2}}, \tag{4.31}$$

with some initial conditions $t(0) = t_0$ and $w(0) = w_0$. Finally the energy density on the brane is read off from the isotropic junction condition,

$$\frac{\kappa^2}{6}\rho = e^{-A_0}\dot{A}\frac{f}{\sqrt{1-f^2}} + \frac{A'}{\sqrt{1-f^2}}. \tag{4.32}$$

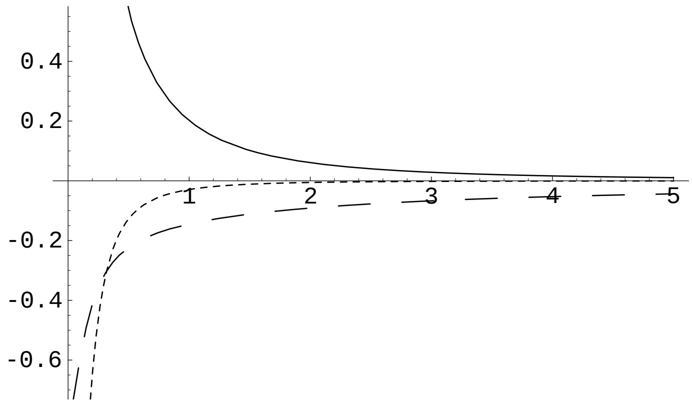

Figure 1: The effective energy density ρ_{eff} (solid line) and pressure P_{eff} (dashed line), in units of σ_{RS}, and the effective equation of state $P_{\text{eff}}/\rho_{\text{eff}}$ (dotted line) as functions of the brane cosmic time τ. The plot corresponds to the initial conditions $t_{\text{b}}(0) = 0.1$ and $w_{\text{b}}(0) = 0.2$.

Notice that ρ is completely determined by the *position* of the brane in the bulk spacetime. In order to get its evolution as a function of τ, one must solve for the trajectory from (4.31) and then substitute into (4.32).

As an illustration, let us consider the case $q_0 = -\sqrt{3}/4$ which corresponds to a brane moving towards increasing values of w according to (4.30) (we implicitly assume that the bulk is endowed with coordinates t and w that are positive). Solving for the cosmological evolution on the brane, one finds that as $\tau \to \infty$, ρ converges towards the Randall-Sundrum tension $\sigma_{\text{RS}} = 6/(\kappa^2 \ell)$. This is shown by the solid line in figure 1 which plots the "effective" energy density $\rho_{\text{eff}} = \rho - \sigma_{\text{RS}}$ as a function of τ. This can be understood by the fact that the bulk geometry, at large w, resembles the warped Kasner geometry (3.23). We have also plotted the effective pressure $P_{\text{eff}} = P + \sigma_{\text{RS}}$, which can be computed from ρ and the average Hubble parameter via the usual energy conservation equation. Observe that the effective pressure is negative but converges towards zero more rapidly than the effective energy, so that the effective equation of state asymptotically corresponds to that of non-relativistic matter.

5. Conclusion

In this paper we have constructed explicit anisotropic brane cosmologies. We began our study by solving the full five-dimensional vacuum Einstein equations with negative cosmological constant for a bulk metric admitting a homogeneous but anisotropic (Bianchi I) three-dimensional slicing. Since the general equations are too difficult, we have specialised our analysis to two particular cases. The first case, in which the bulk anisotropy was assumed to be only time-dependent, lead us to bulk solutions that are *warped versions of 4D vacuum solutions of Einstein equations with a cosmological constant.* These solutions were already known. Then we assumed separability of the metric components into time and extra-dimension dependent pieces. In this way, we obtained, to our knowledge, *new bulk solutions* that combine the 4D Kasner solution and the static 5D solutions of [17].

We then turned to the initial motivation for this work, namely whether or not it is possible to embed an anisotropic brane with only a perfect fluid as matter into a bulk geometry. Somewhat to our surprise, we have found that it is possible to find such a configuration in some of our "hybrid" bulk geometries: this is because the anisotropic pressure on the brane induced by the bulk time anisotropy can be compensated by a corresponding term induced by the bulk spatial anisotropy when the brane is moving. This compensation is possible only for particular trajectories, which implies that there is no longer any flexibility in the choice of the perfect fluid, in contrast with the isotropic case. In some sense, the perfect fluid on the brane is "imposed" by the bulk geometry. In other words, it means that only a very particular type of perfect fluid is compatible with the given bulk geometry.

This limitation is not surprising. It is simply the consequence of having less symmetries. In the isotropic case, we have a high level of symmetry which implies a generalized Birkhoff theorem: because of the symmetries, the motion of the brane cannot perturb the bulk geometry, in the same way as a moving spherical shell cannot generate gravitational waves in 4D Einstein gravity. As soon as we allow for anisotropy on the brane, its motion in the bulk should generate a very complicated bulk. In this respect, it is a rather good surprise that there exists an analytical bulk solution that allows for an anisotropic cosmology in a purely perfect fluid brane.

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A. Parametrising the bulk metric solutions

Our purpose here is to give explicitly the solutions of the constraint equations

$$\sum_i p_i = 1\,, \qquad \sum_i p_i^2 = 1\,, \qquad \sum_\mu q_\mu = 0\,, \qquad \sum_\mu q_\mu^2 = \frac{3}{4}\,, \qquad \sum_i q_i\,(p_i + 1) = 0\,. \tag{A.1}$$

It is possible to check that one can parameterize the solutions of these five constraint equations as follows:

$$p_i = \frac{\sqrt{6}}{3}\,r_i(\phi) + \frac{1}{3}\,, \qquad q_i = \sqrt{\frac{3}{4} - \frac{4}{3}q_0^2}\,r_i(\phi + \theta) - \frac{q_0}{3}\,, \tag{A.2}$$

with

$$r_1(\phi) \;=\; \frac{\sqrt{3}+3}{6}\cos\phi + \frac{\sqrt{3}-3}{6}\sin\phi\,, \tag{A.3}$$

$$r_2(\phi) = \frac{\sqrt{3}-3}{6}\cos\phi + \frac{\sqrt{3}+3}{6}\sin\phi\,, \tag{A.4}$$

$$r_3(\phi) = -\frac{\sqrt{3}}{3}(\cos\phi + \sin\phi)\,, \tag{A.5}$$

and

$$\theta = \cos^{-1}\left(\frac{4\sqrt{2}q_0}{\sqrt{9-16q_0^2}}\right). \tag{A.6}$$

All allowed sets of coefficients are thus expressed in terms of two parameters, q_0 and the angle ϕ.

In the last section, we considered the particular case $q_0 = \pm\sqrt{3}/4$. Substituting in the above results, this yields the following parametrization:

$$p_i = \frac{\sqrt{6}}{3}r_i(\phi) + \frac{1}{3}\,, \qquad q_i = \frac{\sqrt{2}}{2}r_i(\phi+\theta_\pm) \mp \frac{\sqrt{3}}{12}\,, \tag{A.7}$$

with

$$\theta_+ = 0\,, \qquad \theta_- = \pi\,. \tag{A.8}$$

B. Singularities of the bulk metric

In order to analyse the intrinsic properties of a metric, it is useful to construct gauge-invariant quantities. A particular useful quantity in the present context, where the Ricci tensor is already known, is the square of the Weyl tensor.

For the metric (3.43), we find that the square of the bulk Weyl tensor is given by

$$\begin{aligned} C_{ABCD}C^{ABCD} = \frac{1}{3\ell^4}\Bigg\{ & -\frac{8}{9}\ell^4\left(-2+\sqrt{2}\cos(3\phi)-\sqrt{2}\sin(3\phi)\right)z^{-4q_0-1}(1-z^2)t^{-4} + \\ & + \Bigg[\frac{16}{9}q_0^2(-27+56q_0^2)(1-z^2)^2 + 8q_0(27-80q_0^2)(1-z^4) + \\ & \quad + \frac{9}{2}(23+2z^2+23z^4) - \frac{2}{9}\left(9-16q_0^2\right)^{3/2}(1-z^2)\times \\ & \quad \times\left(4q_0(1-z^2)+9(1+z^2)\right)\left(\cos(3\phi+3\theta)-\sin(3\phi+3\theta)\right)\Bigg]\times \\ & \times z^{-4}(1-z^2)^2 - \frac{4}{9}\ell^2\left(9-16q_0^2\right)(1-z^2)\times \\ & \times\left(\sqrt{2}-\cos(3\phi)+\sin(3\phi)\right)z^{-5/2-2q_0}(1-z^2)^{3/2}t^{-2}\Bigg\}. \end{aligned} \tag{B.1}$$

Note that the spacetime becomes singular when $t \to 0$ except when $3\phi + \pi/4 = 0$, which corresponds to one of the p_i equal to 1 while the other two vanish.

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